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Note

On the perfect one-factorization conjecture

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Abstract

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For even n, let $c(n)$ denote the maximum over all one-factorizations $\mathcal{F}$ of K_n of the number of Hamilton cycles obtained by taking pairwise unions of members of $\mathcal{F}$. The perfect one-factorization conjecture is that $c(n) = \binom{n-1}{2}$ for even $n \geq 4$. We show that $c(n) \geq (n-1) \cdot \varphi(n-1)/2$ and give a multiplicative construction which shows that $c(mn+1) \geq 2 \cdot c(m+1) \cdot c(n+1)$ when m and n are odd and relatively prime. Combined with known results this occasionally improves on the first inequality.

A *one-factor* of a graph $G = (V, E)$ is a one-regular spanning subgraph of G. A *one-factorization* of G is a partition of E into one-factors. Obviously, if G has a one-factorization $\mathcal{F} = \{F_1, \ldots, F_d\}$ then G is d-regular. A *perfect pair* of $\mathcal{F}$ is a pair $\{F_k, F_l\}$ such that $F_k \cup F_l$ induces a Hamilton cycle in G. Define $c(\mathcal{F})$ to be the number of perfect pairs of $\mathcal{F}$, and $c(G)$ to be the maximum $c(\mathcal{F})$ over all one-factorizations $\mathcal{F}$ of G.

The perfect one-factorization conjecture is that for $m \geq 2$, $c(K_{2m}) = \binom{2m-1}{2}$, where K_{2m} is the complete graph on $2m$ vertices. In other words, it is conjectured that a one-factorization of K_{2m} exists in which every pair is perfect: such a one-factorization is itself called *perfect*. Several authors have contributed to the literature of this subject since its introduction in [8], notably [1–4, 6, 7, 9, 11, 14, 15]. For a survey of the theory of one-factorizations of K_{2m} in general, see [10], which lists 146 references.

A related problem is to find a set of $\binom{n-1}{2}$ Hamilton cycles in K_n such that each 2-path in K_n is on exactly one of them. When n is even, such a set of Hamilton cycles can be constructed from a perfect one-factorization of K_n. The interested reader can refer to [5, 12, 13] for more information on this problem.

From now on, we will write $c(2m)$ instead of $c(K_{2m})$.

Our proofs can be stated more naturally in terms of near-one-factorizations of K_{2m-1}. A *near-one-factor* of $G=(V, E)$ is a one-factor of $G \setminus v$ for some $v \in V$, and a *near-one-factorization* of G is a partition $\mathscr{F}=\{F_1, \ldots, F_s\}$ of E into near-one-factors. A pair $\{F_k, F_l\}$ in $\mathscr{F}$ is a *perfect pair* if $F_k \cup F_l$ induces a Hamilton path of G. A near-one-factorization is *perfect* if and only if all of its pairs are perfect. We define $c'(\mathscr{F})$ to be the number of perfect pairs of $\mathscr{F}$, and $c'(G)$ to be the maximum $c'(\mathscr{F})$ over all near-one-factorizations $\mathscr{F}$ of G. We also write $c(2m-1)$ for $c'(K_{2m-1})$; since $c(n)$ is defined above only for even numbers, no confusion should result.

It is well known (and easy to see) that K_{2m} has a perfect one-factorization if and only if K_{2m-1} has a perfect near-one-factorization. In fact, the correspondence between one-factorizations of K_{2m} and near-one-factorizations of K_{2m-1} (cf. [10, §1 paragraph 3]) shows that in general $c(2m-1)=c(2m)$.

We collect known results about perfect one-factorizations (and, equivalently, about perfect near-one-factorizations) in Theorem 1, cf. [1–4, 6, 7, 11, 14, 15].

Theorem 1. *For odd n, $c(n)=\binom{n}{2}$ if one of the following conditions holds*:

(a) *n is prime.*

(b) *$n=2p-1$ for a prime p.*

(3) *n is one of* 15, 27, 35, 39, 49, 169, 243, 343, 729, 1234, 1331, 1369, 1849, 2197, 3125, *or* 6859.

The construction which proves part (a) of Theorem 1 is part of the 'folklore' of the perfect one-factorization conjecture, and establishes the following result.

Proposition 2. *For odd n, $c(n) \geqslant n \cdot \varphi(n)/2$, where $\varphi(n)$ is the Euler totient.*

Proof. For $k=0, 1, \ldots, n-1$ let F_k denote the graph on n vertices $1, 2, \ldots, n$ with adjacency matrix A_k defined by

$$(A_k)_{ij}=\begin{cases}1 & \text{if } i \neq j \text{ and } i+j=k \pmod{n},\\ 0 & \text{otherwise.}\end{cases}$$

Then F_k is a near-one-factor of K_n (for $k=0, 1, \ldots, n-1$) and $\mathscr{F}=\{F_0, F_1, \ldots, F_{n-1}\}$ is a near-one-factorization of K_n. It is easy to see that $\{F_k, F_l\}$ is a perfect pair in $\mathscr{F}$ if and only if $k-l$ is relatively prime to n. This shows that $c(\mathscr{F})=n \cdot \varphi(n)/2$, which proves the result. $\square$

The fact that primes figure prominently in Theorem 1 suggests that one could adopt a multiplicative strategy for attacking the perfect one-factorization conjecture. This consists of two parts:

(a) If $n = p^s$ is an odd prime power then $c(n) = \binom{n}{2}$.

(b) If m and n are odd and coprime and $c(m) = \binom{m}{2}$ and $c(n) = \binom{n}{2}$ then $c(mn) = \binom{mn}{2}$.

Theorem 3 is essentially a failed attempt at proving part (b).

Theorem 3. *If m and n are odd and coprime then $c(mn) \geq 2 \cdot c(m) \cdot c(n)$.*

Proof. Let $\mathscr{F}$ be a near-one-factorization of K_m with $c(\mathscr{F}) = c(m)$ and let $\mathscr{G}$ be a near-one-factorization of K_n with $c(\mathscr{G}) = c(n)$. Regard each near-one-factor in $\mathscr{F}$ or $\mathscr{G}$ as having a loop at its isolated vertex. The product of graphs M and N is denoted $M \times N$ and is defined as follows:

$V(M \times N) = V(M) \times V(N)$,

$\{(v, w), (v', w')\} \in E(M \times N)$ if and only if $\{v, v'\} \in E(M)$ and $\{w, w'\} \in E(N)$.

Note that if the complete graph is considered to have a unique loop at each vertex then $K_{mn} \simeq K_m \times K_n$. We construct a near-one-factorization $\mathscr{H}$ of K_{mn} by defining

$$\mathscr{H} = \{F \times G : F \in \mathscr{F} \text{ and } G \in \mathscr{G}\}.$$

It is easy to see that (with the above conventions about loops) $\mathscr{H}$ is indeed a near-one-factorization of K_{mn}. Fig. 1 illustrates typical near-one-factors in $\mathscr{H}$ (with the loops at the isolated vertices omitted). The main step in the proof is the following lemma.

Lemma. *With the notation above, if $\{F, F'\}$ and $\{G, G'\}$ are perfect pairs in $\mathscr{F}$ and $\mathscr{G}$ respectively, then $\{F \times G, F' \times G'\}$ is a perfect pair in $\mathscr{H}$.*

Proof. Index the vertices of K_m by $u_1, \ldots, u_m$ such that the path $F \cup F'$ is $u_1 \cdots u_m$ with u_1 isolated in F' and u_m isolated in F. Similarly, index the vertices

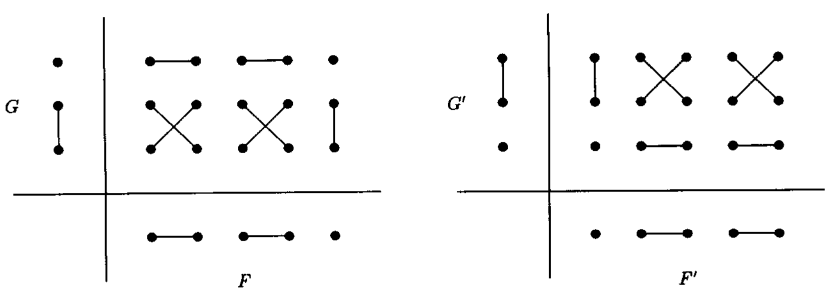

Fig. 1. Near-one-factors $F \times G$ and $F' \times G'$.

of K_n by $v_1, \ldots, v_n$ such that the path $G \cup G'$ is $v_1 \cdots v_n$ with v_1 isolated in G' and v_n isolated in G. The vertices of $K_{mn} \simeq K_m \times K_n$ are thus indexed by $w_{ij} = (u_i, v_j)$ for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ with w_{11} isolated in $F' \times G'$ and w_{mn} isolated in $F \times G$.

Define projection maps π_1 and π_2 on $V(K_{mn})$ by

$$\pi_1(w_{ij}) = i \quad \text{and} \quad \pi_2(w_{ij}) = j.$$

The only vertices of $M = (F \times G) \cup (F' \times G')$ with loops are w_{11} and w_{mn}: these have degree three and all other vertices have degree two. Hence M is the disjoint union of some cycles and a path P with endpoints w_{11} and w_{mn}, each with a loop. We will show that the length of P is at least $mn - 1$ and so $M = P$, proving the lemma.

Suppose that the path P is $p_0 p_1 \cdots p_k$ where $p_0 = w_{11}$ and $p_k = w_{mn}$. Define a path Q in the integer plane $Z \times Z$ by the sequence of vertices:

$$(x_0, y_0) = (0, 0),$$
$$x_t = x_{t-1} + |\pi_1(p_t) - \pi_1(p_{t-1})|,$$
$$y_t = y_{t-1} + |\pi_2(p_t) - \pi_2(p_{t-1})|,$$

for $1 \leqslant t \leqslant k$. Fig. 2 shows the path Q corresponding to the near-one-factors shown in Fig. 1.

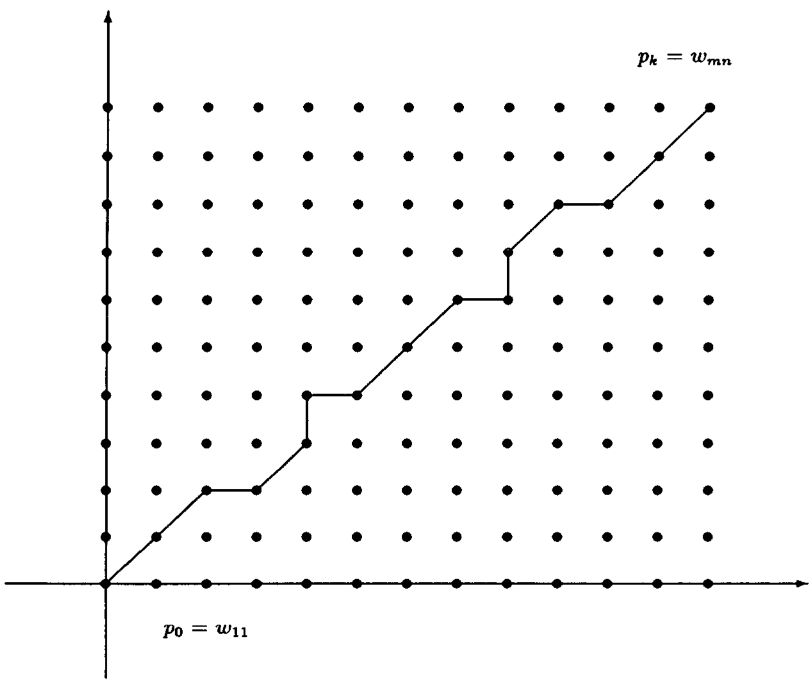

Fig. 2. The path Q.

One notices that in general, for $1 \leqslant t \leqslant k$, $x_t = x_{t-1}$ if and only if m divides t, and $y_t = y_{t-1}$ if and only if n divides t. From this it follows that $m - 1$ divides x_t if and only if $t \equiv 0, -1 \pmod m$, and that $n - 1$ divides y_t if and only if $t \equiv 0, -1 \pmod n$. From the construction it is apparent that $m - 1$ divides x_k and $n - 1$ divides y_k; hence $k \equiv 0, -1 \pmod m$ and $k \equiv 0, -1 \pmod n$. Also from the construction, $x_{k-1} = x_k - 1$ and $y_{k-1} = y_k - 1$, so that m does not divide k and n does not divide k. Therefore m divides $k + 1$ and n divides $k + 1$, and since m and n are coprime, $k + 1 \geqslant mn$, as was to be shown. □

Now the lemma also implies that $\{F \times G', F' \times G\}$ is a perfect pair of $\mathcal{H}$, and it follows that $c(\mathcal{H}) \geqslant 2 \cdot c(\mathcal{F}) \cdot c(\mathcal{G})$. This completes the proof. □

Combined with Theorem 1, Theorem 3 occasionally improves upon the bound of Proposition 2. For example take $n = 231 = 3 \cdot 7 \cdot 11$. Proposition 2 yields $c(231) \geqslant 231 \cdot \frac{120}{2} = 13860$, while from Theorem 3 we obtain $c(231) \geqslant 2 \cdot \binom{33}{2}\binom{7}{2} = 22092$, much closer to the conjectured value of $\binom{231}{2} = 26565$.

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